4.3.1b Appendix B: Portfolio Theory

Unit: Unit 1 – Treasury Strategy

Module: Module 4 – Financial risk

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Summary: Simple assumptions about expected return, standard deviation of returns and correlation between assets allows powerful techniques of optimisation to be adopted

Keywords: portfolio, asset allocation, diversification, random variable, standard deviation, co-variance, correlation, mean-variance

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Learning outcomes:

1. Portfolio analysis deals with collections of assets, where returns on those assets may be linked.
2. There are many applications for the theory in corporate finance and risk management.
3. Relatively simple mathematics allows for portfolio optimisation through mean variance efficiency.

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1 Introduction

Some risks can be best evaluated using techniques of portfolio analysis, especially those where assets in the portfolio interact, or have some correlation. Classic portfolio analysis is concerned with the allocation of capital to investment in a set of securities. Starting with a notional £1, in what proportions should this be divided between the available set of securities? The latter could include equities, bonds, bills, or any other asset. It could even have negative proportions, which are interpreted to mean either borrowing or short selling securities. (Short selling means the process of selling an asset that one does not own to another party on the expectation of being able to buy it back at a lower price.)

In such a context, the measurement of portfolio outcomes is focused on the portfolio return. For every pound of capital invested, a return of +15% is better than a return of +5%, and this will be so, independent of the scale of the portfolio, whether it is £1,000 or £1,000,000 (assuming at this stage no differences in risk). Thus it is common to think in terms of the portfolio division of £1 or $1, and simply multiply up by the required portfolio size in the appropriate currency.

2 The scope of portfolio analysis

Though the early historical development was in the context of equity funds management, the principles of portfolio analysis can be applied to a variety of contexts. Here are some of the more important:

- Management of asset funds, which include equity trusts, investment companies,
- Pension funds, trust funds and companies. The assets concerned are typically equities, money market debt, bonds and other fixed interest securities, mortgages, or real estate. The objective is to maximise the portfolio value; or more specifically, to maximise the holding period returns incorporating current period dividends, coupons etc, in addition to changes in portfolio value. Debt may be used to finance the assets and this should also be factored into the analytical framework.
- The debt portfolio of the firm, comprising a mixture of debt with different maturities, currency and coupons. In this context, the objective is to minimise the economic value of liabilities; in this case, the economic value of the firm's debt. As treasurers are looking ahead to what will happen as debt matures, or as coupon and other payments have to be made, it is common to break down projected cashflows into a series of annual or monthly re-pricing buckets and treat these more or less as though each was a separate asset, or in this case liability.
- The currency portfolio of the firm. This can be either the operating assets, cash, cash flow, debt or earnings of the firm. For example a portfolio approach could be taken with the translation exposure of earnings, comprising say USD, GBP and JPY and the context might be to maximize these in base currency terms. This approach would contrast with tackling each earning stream on its own.
- The firm itself can be viewed as a portfolio, whose elements correspond to payments to various claimants of corporate value. At the most elementary level, these would be the shareholders and the debt holders.

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3 Asset allocation in portfolio analysis

Suppose we have n assets, and let r_i denote the return on asset #i over the holding period, normally expressed in per annum terms. This return includes dividends, coupons, and other cash receipts. It also includes the capital gains or losses as the price of the asset rises or falls over the holding period.

A notional £1 of capital is to be allocated in proportions x_i to asset #i, among n assets or securities. Then the portfolio return per £1 is obtained as the weighted sum of the asset returns:

$$R\rho = x_1 \times r_1 + x_2 \times r_2 + \ldots\ldots + x_n \times r_n = \sum_{1=1}^{n} x_i \times r_i$$

Thus if my initial wealth is W_0 then my end of period wealth is $(1 + R_p) \times W_0$.

One is sometimes tempted to substitute 'value' (V) for 'wealth' and this is fine, but remember that mark to market portfolio value at the end of the period does not always encompass interim cash receipts or payments that may be an important part of portfolio reward. Thus if a value concept is used in terms of a return, it must be in the form of *total return indices* or magnitudes. To compute total returns, we imagine that all dividends, coupons etc. are reinvested in the purchase, at the end of the period, of new units of the asset. The accumulations of the latter are sometimes referred to as *accumulation funds*: the total return is then the growth in the value of this notional fund. The holding period rate of return in fixed interest portfolios assumes such a comparison; similarly, 'gross' stock market indices, as distinct from the lower conventional corresponding 'capital' index (which does not take any account of dividends).

Example 1: Wealth accumulation and total return

An investor buys 100 shares in a company, where the price per share is £20. A year later, a dividend of £1 per share is paid, and the share's ex-dividend price is £25. Immediately afterwards, there is a two-for-one stock split. A year later after the stock split, a dividend of £0.6 per share is paid and the ex-dividend price is £13.50. What are the returns for the two years? (Ignore tax.)

Assume that dividends are declared and paid on the same day. Ignore transaction costs.

Solution

For year 1, the return per share is: $\frac{(25 - 20) + 1}{20} = 0.30 = 30\%$

For year 2 given:

Opening price is £25/2 = £12.50
Closing price is £13.50
Dividends are 60p
The return per share is:

$$\frac{(13.50 - 12.50) + 0.60}{12.50} = 0.1280 = 12.80\%$$

The returns r_i can also refer to returns on sub-portfolios corresponding to different asset classes. Different managers might be chosen to allocate portfolio elements at different levels. Thus in the example below, asset class advisers would manage the overall picture of how much capital was divided between the asset classes, and within each asset class, a specialist manager would divide the allocated capital into individual securities.

Example 2: Asset class diversification

A diversified pension fund might hold assets in the following proportions:

Asset class	**Proportions x_i**
Cash	10%
Commercial bills	5%
Local currency bonds	15%
Offshore bonds	15%
Domestic equities	30%
Offshore equities	15%
Real estate	10%

Within each asset class, there is a portfolio of individual securities, e.g. a selection of FTSE stocks for the equities component, or perhaps just the FTSE 100 index portfolio. A multi-tiered allocation process is also possible. For example, the equities class could itself be subdivided into large cap, mid cap and small cap stocks.

Diversification, in the above arrangement, is obtained at two levels:

- At the macro level, by exposure to the different economic drivers for the asset classes. For example, offshore elements contain exposures to other economies and to foreign exchange. Likewise, the forces that drive real estate are often quite distinct from those that drive equities.
- At the micro level, by exposure to different industries or to different company fortunes within each industry.

Portfolio proportions and their sign

An asset is said to be held *long* if its proportion $x_i > 0$. It is held *short* if $x_i < 0$, i.e. is a negative holding, where x_i is again the amount of capital allocated to asset #i.

With regard to a short position, the position will be 'in the money' if the price of the asset subsequently drops because the repurchase is at a lower price.

A levered equity holding is equivalent to the following:

- a long position in unlevered equity; and
- a short position in debt; or
- a short position in contingent third party claims.

This is typically the implicit portfolio that goes with the purchase of any share and the corporate finance theory about optimization of capital structure is essentially this same question repeated – i.e. what proportion of equity and debt to include in the corporate portfolio.

4 The trade off between risk and expected return

Higher rewards tend to accompany higher risk, hence investment allocations commonly include a trade off between risk and expected reward. (Note the reward we are discussing is the expected return, as return is a random variable. The term random variable means that we cannot predict its value perfectly in advance although we can assign probabilities or expectations to the likely outcomes.)

In exploring this trade off and the role of diversification, it is useful to start with a very simple idea of risk, namely the variance of a portfolio return, i.e. Var [R]; or its standard deviation, conventionally denoted σ (recall that Var[R] = σ^2).

Mean-variance analysis is the name given to the approach which selects and judges portfolios in terms of just two factors: the mean return and the variance of returns. For a given level of risk, a higher mean is preferred; likewise for a given level of return, a lower variance is preferred. One seeks portfolios that maximise the mean, while at the same time minimizing the variance.

5 Portfolio of two or more assets

Suppose that a portfolio with net worth of £1 of capital is to be constructed from two assets: securities, #1 and #2. Let r_1 be the return on the first and r_2 the return on the second, and w_1 and w_2, the weightings, equate to 1. The portfolio return, R_p, is therefore:

$R_p = w_1 \times r_1 + w_2 \times r_2$

The returns r_1 and r_2 are random variables.

Definitions and notation

Means: $\mu_1 = E[r_1]$; $\mu_2 = E[r_2]$ (the prefix E indicates the Expected value for the [variable])

Variances:

$\sigma_1^2 = E[(r_1 - \mu_1)]^2$; for the variance of return #1

$\sigma_2^2 = E[(r_2 - \mu_2)]^2$; for the variance of return #2

Standard deviations:

σ_1 for the standard deviation of return #1

σ_2 for the standard deviation of return #2

(Note: The standard deviation (sometimes abbreviated to STD) is the square root of the variance.)

Co-variance: $\sigma_{12} = E[(r_1 - \mu_1)(r_2 - \mu_2)]$

It is also useful to calculate the co-efficient of correlation between r_1 and r_2:

Correlation co-efficient: $\rho = \frac{\sigma_{12}}{\sigma_1 \sigma_2}$

The correlation co-efficient varies between –1 and +1.

If the correlation co-efficient is positive, it means that positive values of r_1 tend to be associated with positive values of r_2 and vice versa.

If the correlation of co-efficient is negative, it means that positive values of r_1 tend to be associated with negative values of r_2 and vice versa.

The expected return on the portfolio is simply a weighted average of the two expected returns. However, the variance of the portfolio depends upon the covariance of the two returns as well as their individual variances.

$E[R_p] = w_1 \times \mu_1 + w_2 \times \mu_2$

$Var[R_p] = w_1^2 \times \sigma_1^2 + w_2^2 \times \sigma_2^2 + 2 \times w_1 \times w_2 \times \sigma_{12}$

Similarly, for a three asset portfolio:

$E[R_p] = w_1 \times \mu_1 + w_2 \times \mu_2 + w_3 \times \mu_3$

And

$Var[R_p] = w_1^2 \times \sigma_1^2 + w_2^2 \times \sigma_2^2 + w_3^2 \times \sigma_3^2 + 2 \times w_1 \times w_2 \times \sigma_{12} + 2 \times w_1 \times w_3 \times \sigma_{13} + 2 \times w_2 \times w_3 \times \sigma_{23}$

For the n asset portfolio the variance equation becomes (descriptively) the sum of $w_n^2 \times \sigma_n^2$ for each of the n assets, plus the sum of $2 \times w_x \times w_y \times \sigma_{xy}$ for each pair of assets in the portfolio – n(n-1)/2 in all. So for more than a few assets the variance equation becomes time consuming to compute; spreadsheets can be used for small portfolios, but specialist software is needed for larger ones.

By an appropriate choice of portfolio proportions we can achieve of a range of outcomes for the returns or expected returns and variance of the portfolio. The following example illustrates this point.

Example 3a: Portfolio of two securities

Assume two securities, #1 & #2 with the following mean and standard deviations (STDs):

	Mean	STD
Security #1	0.04	0.04
Security #2	0.10	0.10

The weightings are w_1 and w_1, where w_1 relates to the proportion of the portfolio invested in security #1, and w_2 the proportion invested in security 2, the weightings equate to 1.

Calculate mean and STD of the two-security portfolio if the correlation co-efficient is:

ρ = -0.75, +0.75.

For three different scenarios:

Scenario	Security #1 W_1	Security #2 w_2
A	0.7	0.3
B	0.5	0.5
C	0.3	0.7

As $w_2 = 1 - w_1$, we can write the portfolio mean and variance as:

$\mu = w \times \mu_1 + (1 - w) \times \mu_2$

$\sigma^2 = w^2 \times \sigma_1^2 + (1 - w)^2 \times \sigma_2^2 + 2\, w \times (1 - w) \times \rho \times \sigma_1 \times \sigma_2$

Where μ is the mean and σ is the standard deviation of the portfolio (σ^2 is the variance of the portfolio).

Given ρ is not required to calculate the mean of the portfolio, the mean of the portfolio is the same whatever the co-efficient of correlation between the two securities.

Calculating the mean for:

Scenario A: $\mu = 0.7 \times 0.04 + (1 - 0.7) \times 0.10 = 0.058$
Scenario B: $\mu = 0.5 \times 0.04 + (1 - 0.5) \times 0.10 = 0.070$
Scenario C: $\mu = 0.3 \times 0.04 + (1 - 0.3) \times 0.10 = 0.082$

Calculating the variance and STD where ρ = -0.75 for:

Scenario A:
$\sigma^2 = 0.7^2 \times 0.04^2 + (1 - 0.7)^2 \times 0.10^2 + 2 \times 0.7 \times (1 - 0.7) \times (-0.75) \times 0.04 \times 0.10 = 0.000424$

$\sigma = \sqrt{0.000424} = 0.021$

Scenario B:

$\sigma^2 = 0.5^2 \times 0.04^2 + (1 - 0.5)^2 \times 0.10^2 + 2 \times 0.5 \times (1 - 0.5) \times (-0.75) \times 0.04 \times 0.10 = 0.0014$

$\sigma = \sqrt{0.0014} = 0.037$

Scenario C:

$\sigma^2 = 0.3^2 \times 0.04^2 + (1 - 0.3)^2 \times 0.10^2 + 2 \times 0.3 \times (1 - 0.3) \times (-0.75) \times 0.04 \times 0.10 = 0.00378$

$\sigma = \sqrt{0.00378} = 0.062$

The same procedure as above can be used to calculate the variance and STD where ρ = +0.75.

Summarising:

Scenario	**Security #1 W_1**	**Security #2 w_2**	**Portfolio mean**	**Portfolio ρ = -0.75 STD**	**Portfolio ρ = 0.75 STD**
A	0.7	0.3	0.058	0.021	0.054
B	0.5	0.5	0.070	0.037	0.066
C	0.3	0.7	0.082	0.062	0.079

Example 3b: Portfolio of two securities

Excel is useful for computing and plotting the results of choosing different combinations of w between 0 and 1.

Assume two securities, #1 and #2 with the following mean and standard deviations (STDs):

	Mean	**STD**
Security #1	0.04	0.04
Security #2	0.10	0.10

The weightings are w_1 and w_2, where w_1 relates to the proportion of the portfolio invested in security #1, and w_2 the proportion invested in security #2, the weightings equate to 1.

Assuming correlation co-efficients of:

P = -0.75, 0, +1.

An Excel model produced the following combinations of expected returns and STDs:

Table 1: The impact on a two asset portfolio when the weighting is changed

	A	B	C	D	E
1			ρ	ρ	ρ
2	W_1		-0.75	0	1
3	↓	Mean	STD	STD	STD
4	1.50	0.010	0.103	0.078	0.010
5	1.25	0.025	0.071	0.056	0.025
6	1.00	0.040	0.040	0.040	0.040
7	0.90	0.046	0.029	0.037	0.046
8	0.80	0.052	0.022	0.038	0.052
9	0.70	0.058	0.021	0.041	0.058
10	0.60	0.064	0.027	0.047	0.064
11	0.50	0.070	0.037	0.054	0.070
12	0.40	0.076	0.049	0.062	0.076
13	0.30	0.082	0.062	0.071	0.082
14	0.20	0.088	0.074	0.080	0.088
15	0.10	0.094	0.087	0.090	0.094
16	0.00	0.100	0.100	0.100	0.100
17	-0.25	0.115	0.133	0.125	0.115
18	-0.50	0.130	0.166	0.151	0.130

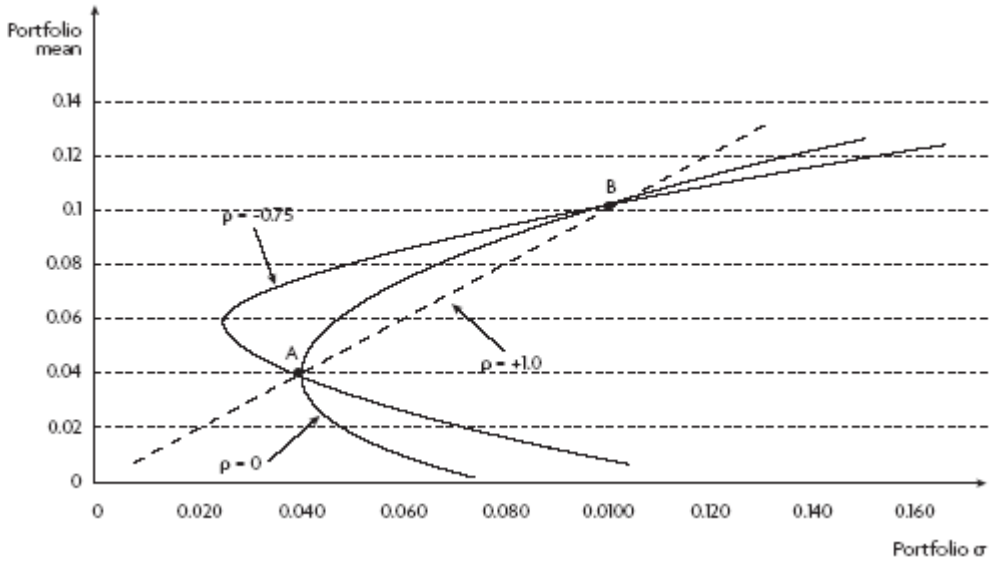
Column A shows the weighting, so for instance a W_1 of 0.3 represents 0.3 of the portfolio is invested in security #1 and 0.7 invested in security #2. Note a weighting of 1.5 represents being long of security #1 by a weighting of 1.5 and being short of security #2 by a weighting of 0.5.

Column B shows the return of the various different weighted portfolios – the calculations for weightings of 0.3, 0.5 and 0.7 agree with the calculations set out in Example 3a.

Columns C, D & E show the STD for the various different weighted portfolios – as above, the calculations for the correlation co-efficients -0.75 and 1 agree with the calculations set out in Example 3a.

The results are graphed below. Conventionally, we put the portfolio sigma on the horizontal axis and the portfolio mean on the vertical.

Exhibit 1: Mean-variance analysis for two securities



Comments:

- Each of the primary securities plots just as a single point. Security#1 plots as point A (with w1 =1) and security #2 as point B (with w1 = 0).
- For any single value of the correlation co-efficient (ρ), there is now an entire set of portfolios obtainable (by varying the proportions of the constituents), each characterised by its own mean and STD.
- For the two asset portfolio where the correlation co-efficient (ρ) = 1 between the two assets, the set of obtainable portfolios fit onto a straight line. (This is because the formula for the standard deviation of the portfolio is w1 x σ1 + (1 – w1) x σ2. In other words the relationship between the standard deviation of the portfolio and the standard deviations of the two securities is linear.)
- Investors have correspondingly greater choice if they choose a combination of series. Consider, for instance, the portfolio curve for ρ = -0.75. This is better than the single security A if the weighting is 0.7 for security #1 and 0.3 for security #2, this is because a 0.7:0.3 weighting has a higher mean and a lower variance.

Once investors have calculated the means, the standard deviations and the co-efficient correlations for a number of assets, they can choose their own preferences as to risk and expected return, depending whether they are risk averse or risk takers.

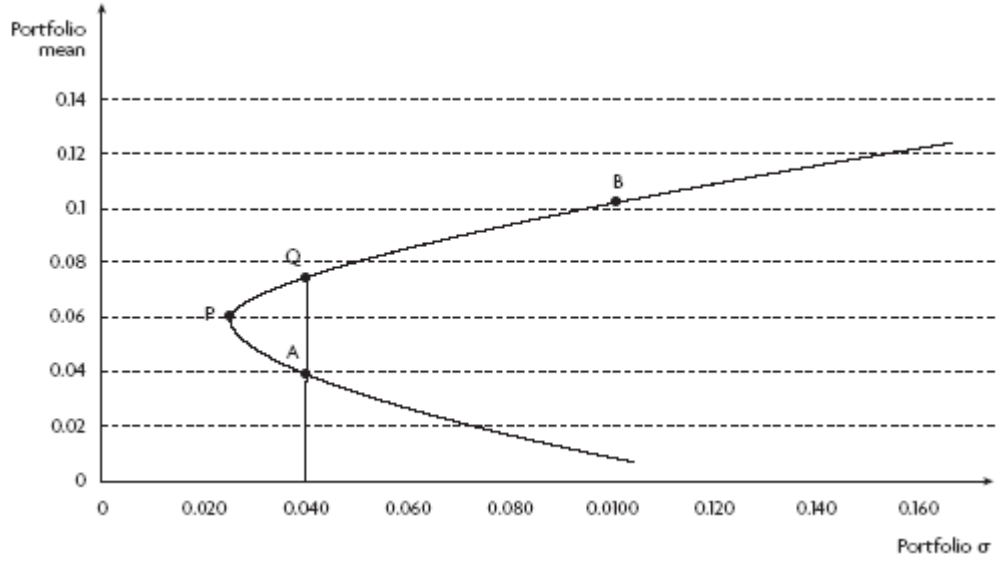
6 Mean-variance efficiency

The *mean-variance efficiency criterion* says one should choose portfolios that:

- maximise the mean for any given variance; and
- minimise the variance for any given mean.

The exhibit below illustrates with two securities where ρ = -0.75, which means that positive values of r_1 tend to be associated with negative values of r_2 and vice versa.

Exhibit 2: Mean-variance analysis for two securities where the co-efficient of correlation is -0.75



The curve APQB represents the set of all attainable portfolios for a given value of the correlation co-efficient between the two basic securities, in this instance ρ = -0.75. Not all of them are efficient. Portfolio Q is clearly better than portfolio A. For the same STD (or variance, or risk) it has a higher expected return.

Proceeding in this way, you can see that the only portfolios that survive such a test are those along the segment PQB. This is called the *efficient frontier* (for the two variable case).

Investors who are risk averse will choose a point like P (the portfolio with minimum σ) and hence the portfolio that generates this expected risk and return. You can find out what this *minimum variance portfolio* is from the above table. The minimum STD is about 0.21 and is associated with a portfolio consisting of 70% of security #1 and 30% of security #2. Less risk averse investors will choose a point north east of P along the efficient frontier, taking on more risk for a higher (average) return.

7 Other decision rules

The problem with the variance as a measure of risk is that it penalises equally positive and negative variations. In effect, you end up trying to cut off both tails of the return distribution, rather than just the negative tail. The value at risk (VaR) criterion addresses this issue – it amounts to defining the risk only in terms of the lower side of the return distribution and disregarding what happens on the upper side.